10 SRF operations

10.1 Introduction

This International Standard specifies operations on SRF coordinates and, in the case of 3D object-spaces, on SRF spatial directions. Underlying these operations is the similarity transformation associated with two ORMs. Similarity transformations are treated first in 10.3. Then the general case of changing the representation of a position as a coordinate in one SRF to its representation as a coordinate in another SRF is specified in 10.4, followed by important special cases. The specification of a spatial direction in the context of an SRF is defined, and the general case of changing the representation of a spatial direction in one SRF to its representation in another SRF is specified (10.5).

Euclidean distance in 2D and 3D object-space is specified in 10.6. Distance and azimuth on the surface of an oblate ellipsoid (or sphere) is specified in 10.7. Vertical offset is defined in 9.3.

10.2 Notation and terminology

An important category of spatial operations is changing spatial information represented in one SRF to spatial information represented in a second SRF. For this category of operations, the adjective “source” shall be used to refer to the first SRF, and the adjective “target” shall be used to refer to the second SRF.

The notation in Table 10.1 is used throughout this clause.

Table 10.1 — Notation

Notation	Description
$\mathrm{ORM_S}$	Source ORM
$\mathrm{ORM_T}$	Target ORM
$\mathrm{ORM_R}$	Reference ORM for a given spatial object
$\boldsymbol{H}_{\mathrm{SR}}$	Reference transformation from $\mathrm{ORM_S}$ to the reference $\mathrm{ORM_R}$
$\boldsymbol{H}_{\mathrm{TR}}$	Reference transformation from $\mathrm{ORM_T}$ to the reference $\mathrm{ORM_R}$
$\boldsymbol{H}_{\mathrm{ST}}$	Similarity transformation from the embedding of $\mathrm{ORM_S}$ to $\mathrm{ORM_T}$
$\mathrm{SRF_S}$	Source SRF based on $\mathrm{ORM_S}$
$\mathrm{SRF_T}$	Target SRF based on $\mathrm{ORM_T}$
$\mathrm{SRF_L}$	The local tangent frame SRF at a coordinate (See 10.5.2)
$\mathrm{CS_S}$	CS of $\mathrm{SRF_S}$
$\mathrm{CS_T}$	CS of $\mathrm{SRF_T}$
$\boldsymbol{G}_{\mathrm{S}}$	Generating function of $\mathrm{CS_S}$
$\boldsymbol{G}_{\mathrm{T}}^{-1}$	Inverse generating function of $\mathrm{CS_T}$
$\boldsymbol{c}_{\mathrm{S}}$	Coordinate of a spatial position in $\mathrm{SRF_S}$
$\boldsymbol{c}_{\mathrm{T}}$	Coordinate of a spatial position in $\mathrm{SRF_T}$
$\boldsymbol{n}_{\mathrm{S}}$	Direction vector in $\mathrm{SRF_S}$ (See 10.5.2)
$\boldsymbol{n}_{\mathrm{T}}$	Direction vector in $\mathrm{SRF_T}$ (See 10.5.2)

10.3 Operations on ORMs

10.3.1 Introduction

The similarity transformations $\boldsymbol{H}_{\text{ST}}$ between source/target pairs ORM_S and ORM_T underlie the coordinate operations in 10.4. Given a set of n ORMs there are $n(n-1)$ such source and target ORM pairs. Instead of specifying the full set of similarity transformations, this International Standard reduces the requirement to specifying the reference transformation $\boldsymbol{H}_{\text{SR}}$ from each object-fixed source ORM_S to the reference ORM_R for a given object. This subclause specifies the methods of expressing a similarity transformation $\boldsymbol{H}_{\text{ST}}$ in terms of the reference transformations for the source and target ORMs. The cases of ORMs for a single object are treated in 10.3.2. The more general cases in which ORM_S and ORM_T are ORMs for different objects are treated in 10.3.3.

10.3.2 ORMs for a single object

If ORM_S is an object-fixed ORM, its reference transformation $\boldsymbol{H}_{\text{SR}}$ may be specified as a seven-parameter transformation in the 3D case (see 7.3.2) and a by four-parameter transformation in the 2D case (see 7.3.3). The general form of $\boldsymbol{H}_{\text{SR}}$ in the 3D case is given by Equation (7). The form in the 2D case is similar. As vector operations, they are in the form of a scaled invertible matrix multiplication followed by a vector addition. This form of vector operation is an invertible affine transformation. In the 3D case using the notation of Equation (5):

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\text{R}} = \boldsymbol{H}_{\text{SR}}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\text{S}}\right) \equiv \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}_{\text{SR}} + \left(1+\Delta s_{\text{SR}}\right)\boldsymbol{T}_{\text{SR}}\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\text{S}} \tag{7}$$

NOTE The processes by which ORMs for the Earth are established are based on physical measurements. These measurements are subject to error and therefore introduce various types of relative distortions between ORMs. Equation (7) is based on the assumption that positions in object-space are error free and the equation includes no compensation for these distortions.

The reference transformation $\boldsymbol{H}_{\text{TR}}$ from ORM_T to ORM_R is similarly specified. An important operation is the similarity transformation $\boldsymbol{H}_{\text{ST}}$ from ORM_S to ORM_T, when neither the source nor the target is necessarily the reference ORM. The $\boldsymbol{H}_{\text{ST}}$ transformation may be expressed as the composition of $\boldsymbol{H}_{\text{SR}}$ with $\boldsymbol{H}_{\text{TR}}^{-1}$ (the inverse of $\boldsymbol{H}_{\text{TR}}$) as in Equation (8) (see Figure 10.1):

$$\boldsymbol{H}_{\text{ST}} = \boldsymbol{H}_{\text{TR}}^{-1} \circ \boldsymbol{H}_{\text{SR}} \tag{8}$$

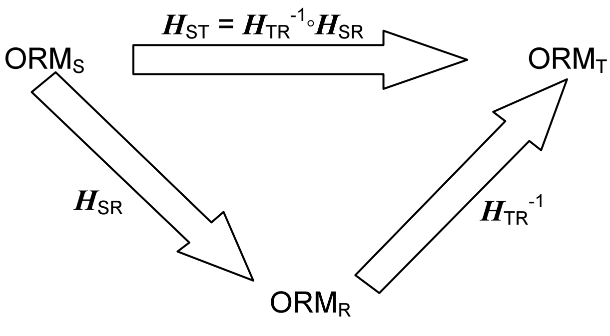

Figure 10.1 — Composed transformations

The inverse operation $\boldsymbol{H}_{\text{TR}}^{-1}$ is also an affine transformation:

$$\boldsymbol{H}_{\mathrm{TR}}^{-1}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\mathrm{R}}\right) = \frac{1}{(1+\Delta s_{\mathrm{TR}})}\boldsymbol{T}_{\mathrm{TR}}^{-1}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\mathrm{R}} - \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}_{\mathrm{ST}}\right)$$
$$= \left(\frac{-1}{(1+\Delta s_{\mathrm{TR}})}\boldsymbol{T}_{\mathrm{TR}}^{-1}\begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}_{\mathrm{TR}}\right) + \frac{1}{(1+\Delta s_{\mathrm{TR}})}\boldsymbol{T}_{\mathrm{TR}}^{-1}\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\mathrm{R}}$$
$$= \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}_{\mathrm{RT}} + \frac{1}{(1+\Delta s_{\mathrm{TR}})}\boldsymbol{T}_{\mathrm{TR}}^{-1}\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\mathrm{R}}$$

Because the matrix $\boldsymbol{T}_{\mathrm{TR}}$ is a rotation matrix, its transpose $\boldsymbol{T}_{\mathrm{TR}}^{\boldsymbol{T}}$ is also its inverse $\boldsymbol{T}_{\mathrm{TR}}^{-1}$. Its inverse is also the matrix $\boldsymbol{T}_{\mathrm{RT}}$ corresponding to the reverse rotations of $\mathrm{ORM}_{\mathrm{T}}$ with respect to $\mathrm{ORM}_{\mathrm{R}}$. In particular:

$$\boldsymbol{T}_{\mathrm{RT}} = \boldsymbol{T}_{\mathrm{TR}}^{-1} = \boldsymbol{T}_{\mathrm{TR}}^{\boldsymbol{T}}$$

and

$$\boldsymbol{H}_{\mathrm{TR}}^{-1}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\mathrm{R}}\right) = \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}_{\mathrm{RT}} + \frac{1}{(1+\Delta s_{\mathrm{TR}})}\boldsymbol{T}_{\mathrm{RT}}\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\mathrm{R}}.$$

The composite operation $\boldsymbol{H}_{\mathrm{ST}} = \boldsymbol{H}_{\mathrm{TR}}^{-1} \circ \boldsymbol{H}_{\mathrm{SR}}$ reduces to:

$$\boldsymbol{H}_{\mathrm{ST}}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\mathrm{S}}\right) = \boldsymbol{H}_{\mathrm{TR}}^{-1} \circ \boldsymbol{H}_{\mathrm{SR}}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\mathrm{S}}\right) \tag{9}$$
$$= \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}_{\mathrm{ST}} + \frac{(1+\Delta s_{\mathrm{SR}})}{(1+\Delta s_{\mathrm{TR}})}\boldsymbol{T}_{\mathrm{ST}}\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\mathrm{S}}$$

where:

$\boldsymbol{T}_{\mathrm{ST}} = \boldsymbol{T}_{\mathrm{RT}}\boldsymbol{T}_{\mathrm{SR}}$, and

$$\begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}_{\mathrm{ST}} = \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}_{\mathrm{RT}} + \frac{1}{(1+\Delta s_{\mathrm{TR}})}\boldsymbol{T}_{\mathrm{RT}}\begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}_{\mathrm{SR}}.$$

If the rotation parameters are equal, then $\boldsymbol{T}_{\mathrm{ST}}$ is the identity matrix, and if $\Delta s_{\mathrm{R}} = \Delta s_{\mathrm{T}}$, $\boldsymbol{H}_{\mathrm{ST}}$ simplifies to a translation of the origin:

$$\boldsymbol{H}_{\mathrm{ST}}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\mathrm{S}}\right) = \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}_{\mathrm{ST}} + \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\mathrm{S}}.$$

Equation (8) and Figure 10.1 also apply to the 2D case.

If the source $\mathrm{ORM}_{\mathrm{S}}$ is a time-dependent ORM for a spatial object, $\mathrm{ORM}_{\mathrm{S}}(t)$ shall denote the $\mathrm{ORM}_{\mathrm{S}}$ at time t, and $\boldsymbol{H}_{\mathrm{SR}}(t)$ shall denote the similarity transformation from the embedding of $\mathrm{ORM}_{\mathrm{S}}(t)$ to the embedding of the object-fixed reference $\mathrm{ORM}_{\mathrm{R}}$. If the similarity transformation $\boldsymbol{H}_{\mathrm{SR}}(t)$ can be determined, it is a time-dependent affine transformation. For a fixed value of time t_0, Equation (8) and Figure 10.1 generalize to

$\boldsymbol{H}_{\mathrm{ST}}\left(t_0\right)=\boldsymbol{H}_{\mathrm{TR}}^{-1}\circ\boldsymbol{H}_{\mathrm{SR}}\left(t_0\right)$. The generalizations to a time-dependent target ORM$_T$(t) are $\boldsymbol{H}_{\mathrm{ST}}\left(t_0\right)=\boldsymbol{H}_{\mathrm{TR}}^{-1}\left(t_0\right)\circ\boldsymbol{H}_{\mathrm{SR}}$ and $\boldsymbol{H}_{\mathrm{ST}}\left(t_0\right)=\boldsymbol{H}_{\mathrm{TR}}^{-1}\left(t_0\right)\circ\boldsymbol{H}_{\mathrm{SR}}\left(t_0\right)$ for the ORM$_S$ static and time-dependent cases, respectively.

EXAMPLE ORM$_S$(t) is the ORM EARTH_INERTIAL_J2000r0 at time t. ORM$_R$ is the Earth reference ORM WGS_1984. Because ORM$_S$(t) and ORM$_R$ share the same embedding origin, the $\boldsymbol{H}_{\mathrm{SR}}(t)$ transformation is a (rotation) matrix multiplication operation (without vector addition). The matrix coefficients for selected values of t account for polar motion, Earth rotation, nutation, and precession. Predicted values for these coefficients are computed and updated weekly by the International Earth Rotation Service (IERS) [IERS] (see 7.5.2). See Annex B for other examples of dynamic ORM reference transformations.

10.3.3 Relating ORMs for different objects

If a spatial object ***S*** exists in the space of another spatial object ***T***, and if ORM$_R$ is the reference ORM for object ***T***, and if the two objects are fixed with respect to each other, then $\boldsymbol{H}_{\mathrm{SR}}$ shall denote a similarity transformation from the embedding of ORM$_S$ to the embedding of ORM$_R$. $\boldsymbol{H}_{\mathrm{SR}}$ is an affine transformation. If ORM$_T$ is an object-fixed ORM for the object ***T*** then $\boldsymbol{H}_{\mathrm{ST}}$ is given by Equation (8). The time dependent generalizations of Equation (8), defined in 10.3.2, are also applicable to this case.

EXAMPLE ORM$_S$ is an ORM for the space shuttle (as a spatial object). ORM$_R$ is the Earth reference ORM WGS_1984. When in orbit at time t, $\boldsymbol{H}_{\mathrm{SR}}(t)$ transforms positions with respect to ORM$_S$ to positions with respect to ORM WGS_1984.

If the object-space of ***S*** and the object-space of ***T*** do not share locations or are otherwise unrelated, a similarity transformation between ORMs for the respective object-spaces is not defined. An abstract object ***S*** and a physical object ***T*** is an important instance of this case (see 10.4.6). However, if $\boldsymbol{H}_{\mathrm{SR}}$ is an invertible affine transformation between ORM$_S$ and the reference ORM for ***T***, then, given an object-fixed ORM for object ***T,*** ORM$_T$, Equation (8) may be used to define an invertible affine transformation $\boldsymbol{H}_{\mathrm{ST}}$, from ORM$_S$ to ORM$_T$.

10.4 Operations to change spatial coordinates between SRFs

10.4.1 Introduction

Given a coordinate c_{S} in a source SRF, SRF$_S$, and a target SRF, SRF$_T$, the change coordinate SRF operation[22] computes the corresponding coordinate c_{T} in SRF$_T$. The general case of changing the spatial coordinate of a location from SRF$_S$ to SRF$_T$ is presented in formulations in 10.4.2 for time-independent (static) and time-dependent ORM relationships. The general case assumes that the source coordinate corresponds to a location that exists in both the source and target object spaces.

In the general case, ORM$_S$ and ORM$_T$ may differ, and the coordinate systems, CS$_S$ and CS$_T$, may differ. The formulation simplifies in the special case[23] for which ORM$_S$ = ORM$_T$ or, more generally, in the case for which the associated normal embeddings match. This case is presented in 10.4.3. In a further specialization of the ORM$_S$ = ORM$_T$ case, it is assumed that CS$_S$ and CS$_T$ are geodetic and/or map projection CSs. These assumptions produce further simplifications (see 10.4.4).

The case for which CS$_S$ = CS$_T$ and ORM$_S$ and ORM$_T$ differ[24] does not generally produce a computational simplification of the general case. However, when both the source and target SRFs are based on the CS LOCOCENTRIC_EUCLIDEAN_3D, a simplification is produced and is presented in 10.4.5. This case is important for operations on directions (10.5.4).

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[22] ISO 19111 defines this case as a coordinate operation.

[23] ISO 19111 defines this case as a coordinate conversion.

[24] ISO 19111 defines this case as a coordinate transformation.

An extension of the change SRF operation to the case of unrelated source and target object-spaces is presented in 10.4.6 for linear SRFs. In that case, the ORM transformation is only restricted to an invertible affine transformation.

10.4.2 Change coordinate SRF operation

$\mathrm{SRF_S}$ and $\mathrm{SRF_T}$ are two object-fixed SRFs for a spatial object and $\boldsymbol{p}$ is a point in object-space that is in the coordinate system domains for both SRFs. $\boldsymbol{c}_\mathrm{S}$ denotes the coordinate of $\boldsymbol{p}$ in $\mathrm{SRF_S}$, and $\boldsymbol{c}_\mathrm{T}$ denotes the coordinate of $\boldsymbol{p}$ in $\mathrm{SRF_T}$. The determination of $\boldsymbol{c}_\mathrm{T}$ as a function of $\boldsymbol{c}_\mathrm{S}$ is an operation on the SRF pair ($\mathrm{SRF_S}$, $\mathrm{SRF_T}$). The most general form of the operation is:

$$\boldsymbol{c}_\mathrm{T} = \boldsymbol{G}_\mathrm{T}^{-1} \circ \boldsymbol{H}_\mathrm{ST} \circ \boldsymbol{G}_\mathrm{S}\left(\boldsymbol{c}_\mathrm{S}\right) \tag{10}$$

where:

- $\boldsymbol{G}_\mathrm{S}$ is the CS generating function of $\mathrm{SRF_S}$,
- $\boldsymbol{H}_\mathrm{ST}$ is the embedding transformation from $\mathrm{ORM_S}$ to $\mathrm{ORM_T}$, and
- $\boldsymbol{G}_\mathrm{T}$ is the CS generating function of $\mathrm{SRF_T}$.

See Figure 10.2. CS generating and inverse generation functions are specified in Clause 5.

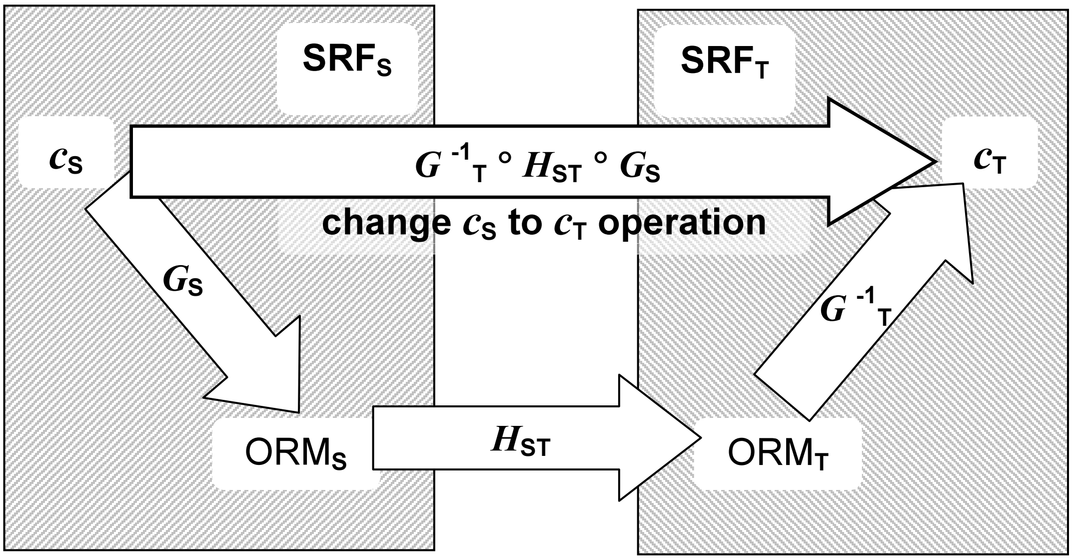

Figure 10.2 — Change coordinate SRF operation

Equation (10) is known as the *Helmert transformation* when $\boldsymbol{H}_\mathrm{ST}$ is approximated with the Bursa-Wolfe equation (see Annex B).

In the time-dependent case, Equation (10) may be generalized to:

$$\boldsymbol{c}_\mathrm{T}\left(t\right) = \boldsymbol{G}_\mathrm{T}^{-1} \circ \boldsymbol{H}_\mathrm{ST}\left(t\right) \circ \boldsymbol{G}_\mathrm{S}\left(\boldsymbol{c}_\mathrm{S}\right).$$

EXAMPLE 1 If $\mathrm{SRF_S}$ and $\mathrm{SRF_T}$ are two celestiodetic SRFs for the same spatial object with different ellipsoid RDs, Equation (10) transforms the coordinate $\boldsymbol{c}_\mathrm{S} = \left(\lambda_\mathrm{S}, \varphi_\mathrm{S}, h_\mathrm{S}\right)$ with respect to one oblate ellipsoid to $\boldsymbol{c}_\mathrm{T} = \left(\lambda_\mathrm{T}, \varphi_\mathrm{T}, h_\mathrm{T}\right)$ with respect to the other oblate ellipsoid.

NOTE A transformation between two celestiodetic SRFs for the spatial object Earth is known as a *horizontal datum shift*. A number of numerical approximations developed to implement this operation have been published. Under the assumption of zero rotations and no scale differences ($\omega_1 = \omega_2 = \omega_3 = 0$ and $\Delta s = 0$), a widely used approximation[25] to directly transform $\boldsymbol{c}_{\mathrm{S}} = (\lambda_{\mathrm{S}}, \varphi_{\mathrm{S}}, h_{\mathrm{S}})$ to $\boldsymbol{c}_{\mathrm{T}} = (\lambda_{\mathrm{T}}, \varphi_{\mathrm{T}}, h_{\mathrm{T}})$, is the *standard Molodensky transformation* formula [83502T] as follows:

$$\begin{pmatrix} \lambda \\ \varphi \\ h \end{pmatrix}_{\mathrm{T}} = \begin{pmatrix} \lambda \\ \varphi \\ h \end{pmatrix}_{\mathrm{S}} + \begin{pmatrix} \Delta\lambda \\ \Delta\varphi \\ \Delta h \end{pmatrix}$$

where:

$$\Delta\lambda = \frac{-\Delta x \sin\lambda + \Delta y \cos\lambda}{\left(R_{\mathrm{N}}(\varphi) + h\right)\cos\varphi}$$

$$\Delta\varphi = \left(\frac{1}{R_{\mathrm{N}}(\varphi) + h}\right)\left\{\begin{aligned} &-\Delta x \sin\varphi\cos\lambda - \Delta y \sin\varphi \sin\lambda + \Delta z \cos\varphi \\ &\quad + \Delta a\left(R_{\mathrm{N}}(\varphi)\varepsilon^2 \sin\varphi\cos\varphi\right)/a \\ &\qquad + \Delta f\left(R_{\mathrm{N}}(\varphi)(a/b)R_{\mathrm{M}}(\varphi)(b/a)\right)\sin\varphi\cos\varphi \end{aligned}\right\}$$

$$\Delta h = \left\{\begin{aligned} &\Delta x\cos\varphi\cos\lambda + \Delta y \cos\varphi\sin\lambda + \Delta z \sin\varphi \\ &\quad - \Delta a\left(a/R_{\mathrm{N}}(\varphi)\right) + \Delta f\,(b/a)R_{\mathrm{N}}(\varphi)\sin^2\varphi \end{aligned}\right\}$$

$\Delta a =$ difference in ellipsoid major semi-axis from source to target

$\Delta f =$ difference in ellipsoid flattening from source to target

The quantities a, b, ε^2, $R_{\mathrm{N}}(\varphi)$, and $R_{\mathrm{M}}(\varphi)$ are defined in Table 5.6.

Equation (10) is only defined for a value of $\boldsymbol{c}_{\mathrm{S}}$ in the CS$_{\mathrm{S}}$ domain if its corresponding position belongs to the CS$_{\mathrm{T}}$ range. If D_{S}^{-1} is the domain of the inverse generating function $\boldsymbol{G}_{\mathrm{S}}^{-1}$ and D_{T}^{-1} is the domain of the inverse generating function $\boldsymbol{G}_{\mathrm{T}}^{-1}$, Equation (10) is only defined for $\boldsymbol{c}_{\mathrm{S}}$ in the set:

$$\boldsymbol{G}_{\mathrm{S}}^{-1}\left(D_{\mathrm{S}}^{-1} \cap \boldsymbol{H}_{\mathrm{ST}}^{-1}\left(D_{\mathrm{T}}^{-1}\right)\right) \equiv \left\{\boldsymbol{c}_{\mathrm{S}} \text{ in } D_{\mathrm{S}} \mid \boldsymbol{H}_{\mathrm{ST}}\left(\boldsymbol{G}_{\mathrm{S}}(\boldsymbol{c}_{\mathrm{S}})\right) \text{ in } D_{\mathrm{T}}^{-1}\right\} \tag{11}$$

EXAMPLE 2 SRF$_{\mathrm{S}}$ is SRF GEOCENTRIC_WGS_1984 and SRF$_{\mathrm{T}}$ is an instance of SRFT MERCATOR, with ORM WGS_1984. Equation (10) is not defined for any $\boldsymbol{c}_{\mathrm{S}}$ that is on the z-axis of SRF$_{\mathrm{S}}$, because the z-axis is not contained in the set in Equation (11).

SRF$_{\mathrm{T}}$ may optionally specify a valid-region V_{T} and may optionally specify an extended-valid region E_{T} (see 8.3.2.4). If D_{T} is the domain of the generating function $\boldsymbol{G}_{\mathrm{T}}$, then $V_{\mathrm{T}} \subseteq E_{\mathrm{T}} \subseteq D_{\mathrm{T}}$. If Equation (10) is defined for $\boldsymbol{c}_{\mathrm{S}}$, $\boldsymbol{c}_{\mathrm{T}}$ may be valid ($\boldsymbol{c}_{\mathrm{T}}$ is in V_{T}), or extended valid ($\boldsymbol{c}_{\mathrm{T}}$ is in $E_{\mathrm{T}} \setminus V_{\mathrm{T}}$) or neither. The set of $\boldsymbol{c}_{\mathrm{S}}$ coordinates for which $\boldsymbol{c}_{\mathrm{T}}$ is valid is:

$$\boldsymbol{G}_{\mathrm{S}}^{-1}\left(D_{\mathrm{S}}^{-1} \cap \boldsymbol{H}_{\mathrm{ST}}^{-1}\left(\boldsymbol{G}_{\mathrm{T}}(E_{\mathrm{T}})\right)\right) \equiv \left\{\boldsymbol{c}_{\mathrm{S}} \text{ in } D_{\mathrm{S}} \mid \boldsymbol{H}_{\mathrm{ST}}\left(\boldsymbol{G}_{\mathrm{S}}(\boldsymbol{c}_{\mathrm{S}})\right) \text{ in } \boldsymbol{G}_{\mathrm{T}}(E_{\mathrm{T}})\right\}$$

where:

$$\boldsymbol{G}_{\mathrm{T}}(E_{\mathrm{T}}) \equiv \left\{\boldsymbol{p} \text{ in } D_{\mathrm{T}}^{-1} \mid \boldsymbol{G}_{\mathrm{T}}^{-1}(\boldsymbol{p}) \text{ in } E_{\mathrm{T}}\right\}.$$

In applications that functionally conform to an SRM profile, the domain of an SRF operation is restricted to the accuracy domain of the SRF as specified by that profile (see Clause 12).

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25 Historically it was thought that these approximations would require less computation than direct conversion. The perceived computational advantage may have been overcome by technology advances. New efficient algorithms for converting celestiocentric coordinates to celestiodetic coordinates have been developed that result in appreciably faster transformations without the attendant loss of accuracy.

10.4.3 The matched normal embeddings case

If both ORMs are the same[23], or, more generally, if the corresponding parameters of the seven-parameter reference transformations of $\mathrm{ORM_S}$ and $\mathrm{ORM_T}$ match, $\boldsymbol{H}_{\mathrm{ST}}$ is the identity transformation. Consequently, Equation (10) simplifies to:

$$\boldsymbol{c}_{\mathrm{T}} = \boldsymbol{G}_{\mathrm{T}}^{-1} \circ \boldsymbol{G}_{\mathrm{S}}\left(\boldsymbol{c}_{\mathrm{S}}\right). \tag{12}$$

EXAMPLE 1 If $\mathrm{SRF_S}$ is a celestiodetic SRF (see 8.4) and $\mathrm{SRF_T}$ is the celestiocentric SRF for the same ORM ($\mathrm{ORM_S} = \mathrm{ORM_T}$), then $\boldsymbol{G}_{\mathrm{T}}^{-1}$ is the identity and Equation (12) reduces to the geodetic generating function: $\boldsymbol{c}_{\mathrm{T}} = \boldsymbol{G}_{\mathrm{S}}\left(\boldsymbol{c}_{\mathrm{S}}\right)$.

EXAMPLE 2 If $\mathrm{SRF_S}$ is an induced surface celestiodetic SRF (see 8.4) and $\mathrm{SRF_T}$ is the 3D celestiodetic SRF for the same ORM ($\mathrm{ORM_S} = \mathrm{ORM_T}$), Equation (12) changes $\boldsymbol{c}_{\mathrm{S}} = (\lambda, \varphi)$ from a coordinate of CS type surface to $\boldsymbol{c}_{\mathrm{T}} = (\lambda, \varphi, 0)$ a coordinate of CS type 3D.

If $\mathrm{SRF_T}$ is a 3D SRF that has ellipsoidal height designated as the vertical coordinate-component of the SRF (see 8.4), and $\mathrm{SRF_S}$ is the induced zero height surface SRF, the *promotion operation* converts a surface coordinate $\boldsymbol{c}_{\mathrm{S}}$ in $\mathrm{SRF_S}$ to a 3D coordinate in $\mathrm{SRF_T}$ by setting the 1st and 2nd coordinate-components of $\boldsymbol{c}_{\mathrm{T}}$ to the 1st and 2nd coordinate-components of $\boldsymbol{c}_{\mathrm{S}}$ and setting the 3rd coordinate-component, ellipsoidal height, to 0. Coordinate promotion is a special case of Equation (12). Applicable SRFs include those based on SRFT CELESTIODETIC, PLANETODETIC, and all map projection SRFTs

EXAMPLE 3 Reversing the roles of source and target SRFs in Example 2, if $\mathrm{SRF_S}$ is a celestiodetic 3D SRF and $\mathrm{SRF_T}$ is the (induced) surface celestiodetic SRF for the same ORM, Equation (12) is not defined for $\boldsymbol{c}_{\mathrm{S}} = (\lambda, \varphi, h)$, unless $h = 0$. Equivalently, only coordinates of the form $\boldsymbol{c}_{\mathrm{S}} = (\lambda, \varphi, 0)$ belong to the set in Equation (11). Coordinates in $\mathrm{SRF_S}$ that are not on the oblate ellipsoid (or sphere) RD instance surface, can be projected to the surface along a coordinate curve by setting $h = 0$.

If $\mathrm{SRF_S}$ is a 3D SRF that has ellipsoidal height designated as the vertical coordinate-component of the SRF (see 8.4), and $\mathrm{SRF_T}$ is the induced zero height surface SRF, the *truncation operation* converts a 3D coordinate $\boldsymbol{c}_{\mathrm{S}}$ in $\mathrm{SRF_S}$ to a surface coordinate $\boldsymbol{c}_{\mathrm{T}}$, by setting the 1st and 2nd coordinate-components of $\boldsymbol{c}_{\mathrm{T}}$ to the 1st and 2nd coordinate-components of $\boldsymbol{c}_{\mathrm{S}}$. The point in object-space corresponding to $\boldsymbol{c}_{\mathrm{S}}$ and the point in object-space corresponding to $\boldsymbol{c}_{\mathrm{T}}$ are not the same point unless $h = 0$. Truncation, therefore, does not generally preserve location.

10.4.4 Map projection SRF and celestiodetic SRF with matched normal embeddings case

The CS generating function $\boldsymbol{G}_{\mathrm{MP}}$ for a map projection SRF (or, respectively, an augmented map projection SRF) is implicitly defined (see 5.8.2 or, respectively, 5.8.6) by the composition of the generating function for the surface geodetic CS (respectively, the geodetic 3D CS) $\boldsymbol{G}_{\mathrm{GD}}$ with the inverse mapping equations $\boldsymbol{Q} \equiv (Q_1, Q_2)$ (respectively, $\boldsymbol{Q} \equiv (Q_1, Q_2, h)$) as:

$$\boldsymbol{G}_{\mathrm{MP}} = \boldsymbol{G}_{\mathrm{GD}} \circ \boldsymbol{Q}.$$

If $\mathrm{SRF_S}$ and $\mathrm{SRF_T}$ are map projection SRFs for the same object, and the corresponding seven parameters of their reference transformations match, then Equation (12) becomes:

$$\begin{aligned} \boldsymbol{c}_{\mathrm{T}} &= \left(\boldsymbol{G}_{GD,\,\mathrm{T}} \circ \boldsymbol{Q}_{\mathrm{T}}\right)^{-1} \circ \left(\boldsymbol{G}_{GD,\,\mathrm{S}} \circ \boldsymbol{Q}_{\mathrm{S}}\right)\left(\boldsymbol{c}_{\mathrm{S}}\right) \\ &= \boldsymbol{P}_{\mathrm{T}} \circ \boldsymbol{G}_{GD,\,\mathrm{T}}^{-1} \circ \boldsymbol{G}_{GD,\,\mathrm{S}} \circ \boldsymbol{Q}_{\mathrm{S}}\left(\boldsymbol{c}_{\mathrm{S}}\right) \end{aligned} \tag{13}$$

for:

$\boldsymbol{Q}_{\mathrm{S}}$: inverse mapping equations for $\mathrm{SRF}_{\mathrm{S}}$,

$\boldsymbol{G}_{GD,\,\mathrm{S}}$: generating function for the surface geodetic (respectively, the geodetic 3D) CS for $\mathrm{SRF}_{\mathrm{S}}$,

$\boldsymbol{Q}_{\mathrm{T}}$: inverse mapping equations for $\mathrm{SRF}_{\mathrm{T}}$,

$\boldsymbol{P}_{\mathrm{T}}$: mapping equations for $\mathrm{SRF}_{\mathrm{T}}$, and

$\boldsymbol{G}_{GD,\,\mathrm{T}}$: generating function for the surface geodetic (respectively, the geodetic 3D) CS for $\mathrm{SRF}_{\mathrm{T}}$.

Furthermore, if $\mathrm{ORM}_{\mathrm{S}} = \mathrm{ORM}_{\mathrm{T}}$, then $\boldsymbol{G}_{GD,\,\mathrm{S}} = \boldsymbol{G}_{GD,\,\mathrm{T}}$ and Equation (13) simplifies to:

$$\boldsymbol{c}_{\mathrm{T}} = \boldsymbol{P}_{\mathrm{T}} \circ \boldsymbol{Q}_{\mathrm{S}}\left(\boldsymbol{c}_{\mathrm{S}}\right). \tag{14}$$

NOTE If $\mathrm{SRF}_{\mathrm{S}}$ is a map projection SRF, and $\mathrm{SRF}_{\mathrm{T}}$ is the corresponding augmented map projection SRF based on the same ORM, then Equation (14) is equivalent to the promotion operation (see 10.4.3).

If $\mathrm{SRF}_{\mathrm{T}}$ is a celestiodetic SRF and $\mathrm{ORM}_{\mathrm{T}} = \mathrm{ORM}_{\mathrm{S}}$, Equation (13) simplifies to:

$$\boldsymbol{c}_{\mathrm{T}} = \boldsymbol{Q}_{\mathrm{S}}\left(\boldsymbol{c}_{\mathrm{S}}\right).$$

Similarly, if $\mathrm{SRF}_{\mathrm{S}}$ is a celestiodetic SRF and $\mathrm{ORM}_{\mathrm{T}} = \mathrm{ORM}_{\mathrm{S}}$, Equation (13) simplifies to:

$$\boldsymbol{c}_{\mathrm{T}} = \boldsymbol{P}_{\mathrm{T}}\left(\boldsymbol{c}_{\mathrm{S}}\right).$$

10.4.5 Linear orthonormal 3D SRF to linear orthonormal 3D SRF cases

The special case of source and target SRFs based on the CS LOCOCENTRIC_EUCLIDEAN_3D is important for the treatment of directions (see 10.5). Every linear orthonormal CS may be viewed as an instance of a CS LOCOCENTRIC_EUCLIDEAN_3D. If $\mathrm{SRF}_{\mathrm{S}}$ and $\mathrm{SRF}_{\mathrm{T}}$ are two SRFT LOCOCENTRIC_EUCLIDEAN_3D based SRFs (see Table 8.11), then the SRF pair operation on $\boldsymbol{c}_{\mathrm{S}} = (u, v, w)$ is determined by substituting the CS LOCOCENTRIC_EUCLIDEAN_3D (see Table 5.9) generating function $\boldsymbol{F}_{\mathrm{LE3D}}$ and its inverse $\boldsymbol{F}_{\mathrm{LE3D}}^{-1}$ in Equation (10). If vectors $\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}$ are the CS binding parameters for the SRFT LOCOCENTRIC_EUCLIDEAN_3D based SRF, $\boldsymbol{F}_{\mathrm{LE3D}}$ may be expressed in the form of the affine transformation:

$$\begin{aligned}
\boldsymbol{F}_{\mathrm{LE3D}}(\boldsymbol{c}) = \boldsymbol{F}_{\mathrm{LE3D}}\begin{pmatrix} u \\ v \\ w \end{pmatrix} &= \boldsymbol{q} + u\boldsymbol{r} + v\boldsymbol{s} + w\boldsymbol{t} \\
&= \boldsymbol{q} + u\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} + v\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} + w\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} \\
&= \boldsymbol{q} + \boldsymbol{R}\,\boldsymbol{c}
\end{aligned}$$

where:

$\boldsymbol{t} = (\boldsymbol{r} \times \boldsymbol{s})$, and

$$\boldsymbol{R} = \begin{pmatrix} r_1 & s_1 & t_1 \\ r_2 & s_2 & t_2 \\ r_3 & s_3 & t_3 \end{pmatrix}.$$

The inverse generating function is expressed as:

$$\boldsymbol{F}_{\mathrm{LE3D}}^{-1}\left(\begin{pmatrix} u \\ v \\ w \end{pmatrix}\right) = \boldsymbol{R}^{\mathrm{T}}\left(\begin{pmatrix} u \\ v \\ w \end{pmatrix} - \boldsymbol{q}\right)$$

where: $\boldsymbol{R}^{\mathrm{T}}$ is the transpose of $\boldsymbol{R}$.

If vectors $\boldsymbol{q}_{\mathrm{S}}$, $\boldsymbol{r}_{\mathrm{S}}$, $\boldsymbol{s}_{\mathrm{S}}$, $\boldsymbol{q}_{\mathrm{T}}$, $\boldsymbol{r}_{\mathrm{T}}$, and $\boldsymbol{s}_{\mathrm{T}}$ are the CS binding parameters for $\mathrm{SRF_S}$ and $\mathrm{SRF_T}$ respectively (see Table 8.11), then substituting the expression in Equation (9) for $\boldsymbol{H}_{\mathrm{ST}}$, Equation (10) specializes to:

$$\begin{aligned} \boldsymbol{c}_{\mathrm{T}} &= \boldsymbol{F}^{-1}_{\mathrm{LE3D,\,T}} \circ \boldsymbol{H}_{\mathrm{ST}} \circ \boldsymbol{F}_{\mathrm{LE3D,\,S}}\left(\boldsymbol{c}_{\mathrm{S}}\right) \\ &= \boldsymbol{R}_{\mathrm{T}}^{\mathrm{T}}\left(\boldsymbol{H}_{\mathrm{ST}}\left(\boldsymbol{q}_{\mathrm{S}} + \boldsymbol{R}_{\mathrm{S}}\boldsymbol{c}_{\mathrm{S}}\right) - \boldsymbol{q}_{\mathrm{T}}\right) \\ &= \boldsymbol{R}_{\mathrm{T}}^{\mathrm{T}}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\mathrm{ST}} + \left(\frac{\left(1+\Delta s_{\mathrm{SR}}\right)}{\left(1+\Delta s_{\mathrm{TR}}\right)}\boldsymbol{T}_{\mathrm{ST}}\boldsymbol{q}_{\mathrm{S}}\right) - \boldsymbol{q}_{\mathrm{T}}\right) + \frac{\left(1+\Delta s_{\mathrm{SR}}\right)}{\left(1+\Delta s_{\mathrm{TR}}\right)}\boldsymbol{R}_{\mathrm{T}}^{\mathrm{T}}\boldsymbol{T}_{\mathrm{ST}}\boldsymbol{R}_{\mathrm{S}}\boldsymbol{c}_{\mathrm{S}}. \end{aligned} \tag{15}$$

In the case that the corresponding seven parameters of the reference transformations of $\mathrm{ORM_S}$ and $\mathrm{ORM_T}$ match, Equation (12) specializes to Equation (16):

$$\begin{aligned} \boldsymbol{c}_{\mathrm{T}} &= \boldsymbol{F}^{-1}_{\mathrm{LE3D,\,T}} \circ \boldsymbol{F}_{\mathrm{LE3D,\,S}}\left(\boldsymbol{c}_{\mathrm{S}}\right) \\ &= \boldsymbol{R}_{\mathrm{T}}^{\mathrm{T}}\left(\boldsymbol{q}_{\mathrm{S}} - \boldsymbol{q}_{\mathrm{T}}\right) + \boldsymbol{R}_{\mathrm{T}}^{\mathrm{T}}\boldsymbol{R}_{\mathrm{S}}\boldsymbol{c}_{\mathrm{S}} \end{aligned}. \tag{16}$$

10.4.6 Changing abstract space linear SRF coordinates to a linear SRF in the space of another object

Engineering designs and other abstract models are often intended for realization in the physical world.

EXAMPLE A building plan is designed in the source $\mathrm{SRF_S}$, an abstract space LOCAL_SPACE_RECTANGULAR_3D SRF. A terrestrial site survey establishes the origin of the target $\mathrm{SRT_T}$, a LOCAL_TANGENT_SPACE_EUCLIDEAN SRF. Source coordinates are identified to target coordinates by: $\left(x_{\mathrm{T}}, y_{\mathrm{T}}, z_{\mathrm{T}}\right) = \left(1+\Delta s\right)\left(x_{\mathrm{S}}, y_{\mathrm{S}}, z_{\mathrm{S}}\right)$ where $\left(1+\Delta s\right)$ is a scale factor.

More generally, abstract models are scaled, rotated, or otherwise transformed by an invertible matrix 3x3 $\boldsymbol{W}$ before a source coordinate is identified to a target coordinate. In many application domains, this similarity transformation is in the form:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\mathrm{T}} = \begin{pmatrix} x_{\Delta} \\ y_{\Delta} \\ z_{\Delta} \end{pmatrix} + k\boldsymbol{W}\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\mathrm{S}}$$

where $k = \left(1+\Delta s\right)$ is the scale factor and $\left(x_{\Delta} \quad y_{\Delta} \quad z_{\Delta}\right)$ is the translation displacement vector and $\boldsymbol{W}$ is a rotation matrix. In the computer graphics application domain this transformation is often represented in matrix 4x4 form:

$$\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}_{\mathrm{T}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & x_{\Delta} \\ a_{21} & a_{22} & a_{23} & y_{\Delta} \\ a_{31} & a_{32} & a_{33} & z_{\Delta} \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}_{\mathrm{S}}, \text{ where } \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = k\boldsymbol{W}.$$

This identification that the transformation makes between source and target coordinates may be viewed as a change coordinate SRF operation from c_{S} in $\mathrm{SRF_S}$, an abstract space LOCAL_SPACE_RECTANGULAR_3D SRF, to a coordinate c_{T} in $\mathrm{SRF_T}$, a physical world LOCOCENTRIC_EUCLIDEAN_3D SRF.

In the notation of 10.4.5:

$$G_S(c_S) = R_S c_S, \text{ and}$$
$$G_T(c_T) = q_T + R_T c_T.$$

Define an invertible affine transformation H_{ST} as $H_{ST}(v) = q_T + R_T W v$ (see 10.3.3). Substitute this H_{ST} in Equation (10) and simplify:

$$\begin{aligned} c_T &= G_T^{-1} \circ H_{ST} \circ G_S(c_S) \\ &= R_T^T \left(H_{ST}(R_S c_S) - q_T \right) \\ &= R_T^T \left(q_T + R_T W R_S c_S - q_T \right) \\ &= W R_S c_S \end{aligned} \tag{17}$$

This illustrates that the identification $c_T = W R_S c_S$ may be viewed as a change coordinate SRF operation.

NOTE Equation (17) illustrates that digital graphic composite pattern modelling techniques such as SceneGraph trees that use scale and rotation matrices W together with translation operations at each tree node are special cases of Equation (10). See also 10.5.4 Example 2.

10.5 Spatial directions and change SRF operations on directions

10.5.1 Introduction

In 3D position-space, a direction is unambiguously specified by a unit vector. The direction specified is translation independent. This is illustrated by lines through points in a given direction $\boldsymbol{n}$ (see A.7.1.1 Example 1). All such lines are parallel. This translation invariance carries over to the coordinate-space of a linear CS, but not to other CSs with vector space structure. In particular, an augmented map projection inherits the vector space structure of 3D Euclidean coordinate-space, but the “up pointing” vector $\boldsymbol{n}$ = (0, 0, 1) points in different spatial directions (in position-space) depending on the map coordinate location from which $\boldsymbol{n}$ is viewed.

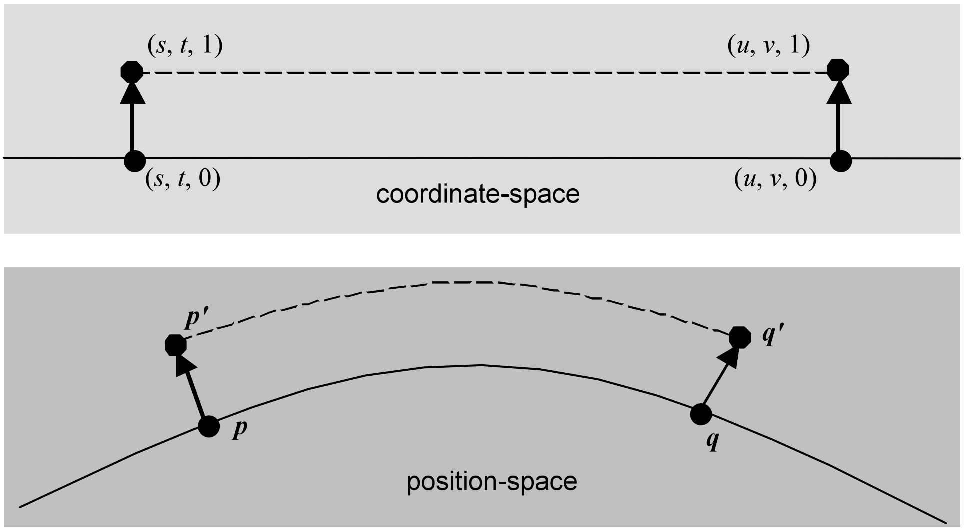

Figure 10.3 — Coordinate-space and position-space directions compared

In Figure 10.3, distinct position points $\boldsymbol{p}$ and $\boldsymbol{q}$ on the ellipsoid surface are projected to augmented map coordinates $(s, t, 0)$ and $(u, v, 0)$. Starting at these map coordinates, the coordinates one unit away in direction $\boldsymbol{n}$ are $(s, t, 1)$ and $(u, v, 1)$ respectively. In an augmented map projection, these coordinates correspond to the position-space points $\boldsymbol{p}'$ and $\boldsymbol{q}'$. The direction from $\boldsymbol{p}$ to $\boldsymbol{p}'$ is not the same as the direction from $\boldsymbol{q}$ to $\boldsymbol{q}'$. It is noted in 5.8.6.2 that augmented map projections are not vertically conformal, therefore angular relationships of spatial directions are generally not preserved by augmented map projections.

A linear CS will not preserve angular relationships between directions unless the CS is also orthonormal. In an SRF based on a linear orthonormal CS, the translation invariant vector space structure of the abstract CS carries over to the spatial CS because the underlying normal embedding preserves angles and distances.

The coordinate-space of a curvilinear CS does not have a linear vector space structure so there is no natural way to specify a translation invariant direction with curvilinear coordinates. An SRF based on a curvilinear CS requires a uniform method for associating a unique linear orthonormal CS based SRF to each coordinate in the interior of the CS domain. This association is defined in 10.5.2.

10.5.2 Specification of direction

In this International Standard, a direction in a 3D orthogonal[26] SRF_S is expressed as a combination of a unit vector and a reference coordinate. The unit vector is in a 3D linear orthonormal SRF, denoted by SRF_L. If SRF_S is curvilinear, SRF_L is uniquely defined for each reference coordinate using the unit tangent vectors to the coordinate-component curves at the reference coordinate. These vectors are used as SRF parameters for SRFT LOCOCENTRIC_EUCLIDEAN_3D with ORM_S to specify SRF_L. SRF_L is termed the local tangent frame at the reference coordinate.

The same definition is applicable if SRF_S is linear. In the linear case, SRF_L at reference coordinate (0, 0, 0) coincides with SRF_S as a spatial CS. Also in the linear case, the unit vector representing the direction is independent of the reference coordinate used. The linear case includes SRFs that are based on SRFTs CELESTIOCENTRIC, LOCAL_TANGENT_SPACE_EUCLIDEAN, LOCOCENTRIC_EUCLIDEAN_3D, and LOCAL_SPACE_RECTANGULAR_3D.

Given a coordinate $\boldsymbol{c} = (u_0, v_0, w_0)$ in the interior of the domain of a 3D orthogonal SRF_S, the *local tangent frame at coordinate* $\boldsymbol{c}$, SRF_L, is the SRF specified by the SRFT LOCOCENTRIC_EUCLIDEAN_3D with ORM_S and the following SRF parameters:

$$\boldsymbol{q} = \boldsymbol{G}(u_0, v_0, w_0), \qquad (18)$$

$$\boldsymbol{r} = \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|}, \text{ and}$$

$$\boldsymbol{s} = \frac{\boldsymbol{v}_2}{\|\boldsymbol{v}_2\|}$$

where:

$$\boldsymbol{v}_1 = \left(\frac{\mathrm{d}\boldsymbol{C}_1}{\mathrm{d}u}\right)_{u=u_0},$$

$$\boldsymbol{v}_2 = \left(\frac{\mathrm{d}\boldsymbol{C}_2}{\mathrm{d}v}\right)_{v=v_0},$$

$\boldsymbol{C}_1$ is the 1st coordinate-component curve at (u_0, v_0, w_0), and

$\boldsymbol{C}_2$ is the 2nd coordinate-component curve at (u_0, v_0, w_0).

The vectors $\boldsymbol{r}$ and $\boldsymbol{s}$ are termed the *local tangent vectors at* $\boldsymbol{c}$. Coordinate-component curves are defined in 5.5.3.

NOTE 1 The tangent vector to the 3rd coordinate-curve at (u_0, v_0, w_0) points in the same direction as the vector $\boldsymbol{t} = \boldsymbol{r} \times \boldsymbol{s}$ because of the coordinate-component ordering restriction specified in 5.6.4.

A *direction* in an orthogonal CS based SRF_S shall be comprised of:

a) a coordinate $\boldsymbol{c}$ in the interior of the CS domain of SRF_S, and

b) a unit vector $\boldsymbol{n}$ in the local tangent frame at $\boldsymbol{c}$.

[26] All of the 3D SRFTs in this International Standard are based on orthogonal CSs.

The coordinate $\boldsymbol{c}$ is termed the *reference coordinate* of the direction and its corresponding position is termed the *reference position* for the direction. The vector $\boldsymbol{n}$ is termed the *direction vector at* $\boldsymbol{c}$.

NOTE 2 The local tangent frame at a coordinate is an instance of the SRFT LOCOCENTRIC_EUCLIDEAN_3D that provides a vector space setting for vector operations on direction vectors at $\boldsymbol{c}$.

EXAMPLE 1 If SRF_S is a LOCOCENTRIC_EUCLIDEAN_3D SRF with SRF parameters $\boldsymbol{q}$, $\boldsymbol{r}$ and $\boldsymbol{s}$, and $\boldsymbol{c}$ is an SRF_S reference coordinate, then local tangent vectors at $\boldsymbol{c}$ are equal to the SRF parameters $\boldsymbol{r}$ and $\boldsymbol{s}$. If $\boldsymbol{c} = (0,0,0)$, then SRF_L = SRF_S.

EXAMPLE 2 SRF_S is an EQUATORIAL_INERTIAL SRF. This SRF is based on the EQUATORIAL_SPHERICAL CS. If $\boldsymbol{c} = (\lambda_0, \theta_0, \rho_0)$ is a reference coordinate, then the local tangent vectors at $\boldsymbol{c}$ are:

$$\boldsymbol{r} = \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|} \text{ and } \boldsymbol{s} = \frac{\boldsymbol{v}_2}{\|\boldsymbol{v}_2\|}$$

where:

$$\begin{aligned}
\boldsymbol{v}_1 &= \left(\frac{\mathrm{d}C_1}{\mathrm{d}\lambda}\right)_{\lambda=\lambda_0} = \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\big(\rho_0\cos(\theta_0)\cos(\lambda),\ \ \rho_0\cos(\theta_0)\sin(\lambda),\ \ \rho_0\sin(\theta_0)\big)\right)_{\lambda=\lambda_0} \\
&= \big(-\rho_0\cos(\theta_0)\sin(\lambda_0),\ \ \rho_0\cos(\theta_0)\cos(\lambda_0),\ \ 0\big), \\
\frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|} &= \big(-\sin(\lambda_0),\ \ \cos(\lambda_0),\ \ 0\big), \\
\boldsymbol{v}_2 &= \left(\frac{\mathrm{d}C_2}{\mathrm{d}\theta}\right)_{\theta=\theta_0} = \left(\frac{\mathrm{d}}{\mathrm{d}\theta}\big(\rho_0\cos(\theta)\cos(\lambda_0),\ \ \rho_0\cos(\theta)\sin(\lambda_0),\ \ \rho_0\sin(\theta)\big)\right)_{\theta=\theta_0} \\
&= \big(-\rho_0\sin(\theta_0)\cos(\lambda_0),\ \ -\rho_0\sin(\theta_0)\sin(\lambda_0),\ \ \rho_0\cos(\theta_0)\big), \text{ and} \\
\frac{\boldsymbol{v}_2}{\|\boldsymbol{v}_2\|} &= \big(-\sin(\theta_0)\cos(\lambda_0),\ \ -\sin(\theta_0)\sin(\lambda_0),\ \ \cos(\theta_0)\big).
\end{aligned}$$

EXAMPLE 3 SRF_S is a CELESTIODETIC SRF. This SRF is based on the GEODETIC CS. If $\boldsymbol{c} = (\lambda_0, \varphi_0, h_0)$ is a reference coordinate, then the local tangent vectors at $\boldsymbol{c}$ are:

$$\boldsymbol{r} = \big(-\sin(\lambda_0),\ \ \cos(\lambda_0),\ \ 0\big), \text{ and } \boldsymbol{s} = \big(-\sin(\varphi_0)\cos(\lambda_0),\ \ -\sin(\varphi_0)\sin(\lambda_0),\ \ \cos(\varphi_0)\big).$$

The vector $\boldsymbol{t} = \boldsymbol{r} \times \boldsymbol{s} = (\cos\lambda_0\cos\varphi_0,\ \ \sin\lambda_0\cos\varphi_0,\ \ \sin\varphi_0)$.

In this example, SRF_L is an LOCAL_TANGENT_SPACE_EUCLIDEAN SRF with SRF parameters $\lambda = \lambda_0$, $\varphi = \varphi_0$, $\alpha = 0$, $x_\text{F} = y_\text{F} = 0$, and h_0.

EXAMPLE 4 SRF_S is based on a conformal map projection CS. If $\boldsymbol{c} = (u_0, v_0, h_0)$ is a reference coordinate, and $(\lambda_0, \varphi_0, h_0)$ is the corresponding celestiodetic coordinate, then the local tangent vectors at $\boldsymbol{c}$ are:

$$\begin{aligned}
\boldsymbol{r} &= \big(-\sin\lambda_0\cos\gamma_0 + \cos\lambda_0\sin\varphi_0\sin\gamma_0,\ \ \cos\lambda_0\cos\gamma_0 + \sin\lambda_0\sin\varphi_0\sin\gamma_0,\ \ -\cos\varphi\sin\gamma_0\big), \text{ and} \\
\boldsymbol{s} &= \big(-\sin\lambda_0\sin\gamma_0 - \cos\lambda_0\sin\varphi_0\cos\gamma_0,\ \ \cos\lambda_0\sin\gamma_0 - \sin\lambda_0\sin\varphi_0\cos\gamma_0,\ \ \cos\varphi_0\cos\gamma_0\big)
\end{aligned}$$

where:

$\gamma_0 = \gamma(\lambda_0, \varphi_0)$ the convergence of the meridian.

In this example, SRF_L is an LOCAL_TANGENT_SPACE_EUCLIDEAN SRF with SRF parameters $\lambda = \lambda_0$, $\varphi = \varphi_0$, $\alpha = \gamma_0$, $x_\text{F} = y_\text{F} = 0$, and h_0.

10.5.3 Changing the reference coordinate of a direction

Given a direction represented with direction vector $\boldsymbol{n}_1$ at $\boldsymbol{c}_1$, the same direction may be represented at another reference coordinate $\boldsymbol{c}_2$ in the same SRF, with direction vector $\boldsymbol{n}_2$. The direction vector $\boldsymbol{n}_2$ is computed as:

$$\boldsymbol{n}_2 = \boldsymbol{R}_2^{\mathrm{T}} \boldsymbol{R}_1 \, \boldsymbol{n}_1 \tag{19}$$

where: for $i = 1, 2$,

$$\boldsymbol{R}_i = \begin{pmatrix} r_{i,1} & s_{i,1} & t_{i,1} \\ r_{i,2} & s_{i,2} & t_{i,2} \\ r_{i,3} & s_{i,3} & t_{i,3} \end{pmatrix},$$

$\boldsymbol{R}_2^{\mathrm{T}}$ = the transpose of $\boldsymbol{R}_2$,

$\boldsymbol{t}_i = \boldsymbol{r}_i \times \boldsymbol{s}_i = \begin{pmatrix} t_{i,1} & t_{i,2} & t_{i,3} \end{pmatrix}$, and

$\boldsymbol{r}_i = \begin{pmatrix} r_{i,1} & r_{i,2} & r_{i,3} \end{pmatrix}$ and $\boldsymbol{s}_i = \begin{pmatrix} s_{i,1} & s_{i,2} & s_{i,3} \end{pmatrix}$ are the local tangent vectors at $\boldsymbol{c}_i$.

The local tangent vectors are computed as in Equation (18). Equation (19) is derived from Equation (16) by dropping the translation term since directions are translation invariant.

If the SRF is based on a linear CS, then the matrix $\boldsymbol{R}_2^{\mathrm{T}} \boldsymbol{R}_1$ in Equation (19) is the identity matrix and $\boldsymbol{n}_1 = \boldsymbol{n}_2$. This implies that in an SRF based on a linear orthonormal CS, a direction vector is independent of the reference coordinate. Thus, Equation (19) is only of interest in the case of a curvilinear SRF.

10.5.4 Changing the SRF representation of a direction

Given a direction represented with direction vector $\boldsymbol{n}_{\mathrm{S}}$ at $\boldsymbol{c}_{\mathrm{S}}$ in SRF$_{\mathrm{S}}$, the same direction may be represented at reference coordinate $\boldsymbol{c}_{\mathrm{T}}$, with direction vector $\boldsymbol{n}_{\mathrm{T}}$ in SRF$_{\mathrm{T}}$. If $\boldsymbol{H}_{\mathrm{ST}}$ is the similarity transformation from ORM$_{\mathrm{S}}$ to ORM$_{\mathrm{T}}$ and $\boldsymbol{T}_{\mathrm{ST}}$ is the matrix in the last term in Equation (15), then the direction vector $\boldsymbol{n}_{\mathrm{T}}$ is computed as:

$$\boldsymbol{n}_{\mathrm{T}} = \boldsymbol{R}_{\mathrm{T}}^{\mathrm{T}} \boldsymbol{T}_{\mathrm{ST}} \boldsymbol{R}_{\mathrm{S}} \, \boldsymbol{n}_{\mathrm{S}} \tag{20}$$

where: for i = S or T,

$$\boldsymbol{R}_i = \begin{pmatrix} r_{i,1} & s_{i,1} & t_{i,1} \\ r_{i,2} & s_{i,2} & t_{i,2} \\ r_{i,3} & s_{i,3} & t_{i,3} \end{pmatrix},$$

$\boldsymbol{R}_{\mathrm{T}}^{\mathrm{T}}$ = the transpose of $\boldsymbol{R}_{\mathrm{T}}$,

$\boldsymbol{t}_i = \boldsymbol{r}_i \times \boldsymbol{s}_i = \begin{pmatrix} t_{i,1} & t_{i,2} & t_{i,3} \end{pmatrix}$, and

$\boldsymbol{r}_i = \begin{pmatrix} r_{i,1} & r_{i,2} & r_{i,3} \end{pmatrix}$ and $\boldsymbol{s}_i = \begin{pmatrix} s_{i,1} & s_{i,2} & s_{i,3} \end{pmatrix}$ are the local tangent vectors at $\boldsymbol{c}_i$.

Equation (20) is derived from Equation (15) by dropping the translation term since directions are translation invariant and dropping the scale factor $(1+\Delta s_{\mathrm{SR}})/(1+\Delta s_{\mathrm{TR}})$ since $\boldsymbol{n}_{\mathrm{T}}$ is a unit vector.

EXAMPLE 1 SRF$_{\mathrm{S}}$ is SRF GEODETIC_WGS_1984 and SRF$_{\mathrm{T}}$ is SRF GEOCENTRIC_WGS_1984. With SRF$_{\mathrm{S}}$ reference coordinate $\boldsymbol{c}_{\mathrm{S}} = (\lambda, \ \varphi, \ 0) = (-77\pi/180, \ +38{,}88\pi/180, \ 0)$, the Washington monument, an obelisk, points approximately in the direction $\boldsymbol{n}_{\mathrm{S}} = (0, \ 0, \ 1)$ at $\boldsymbol{c}_{\mathrm{S}}$. In this example, ORM$_{\mathrm{S}}$ = ORM$_{\mathrm{T}}$ so that $\boldsymbol{T}_{\mathrm{ST}}$ is the identity matrix, and because SRF$_{\mathrm{T}}$ is based on SRFT CELESTIOCENTRIC, $\boldsymbol{R}_{\mathrm{T}}$ is also the identity matrix. Consequently Equation (20) reduces to:

$$\boldsymbol{n}_{\mathrm{T}} = \boldsymbol{R}_{\mathrm{S}} \boldsymbol{n}_{\mathrm{S}} = \begin{pmatrix} r_{\mathrm{S},1} & s_{\mathrm{S},1} & t_{\mathrm{S},1} \\ r_{\mathrm{S},2} & s_{\mathrm{S},2} & t_{\mathrm{S},2} \\ r_{\mathrm{S},3} & s_{\mathrm{S},3} & t_{\mathrm{S},3} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \boldsymbol{t}_{\mathrm{S}}.$$

Then using the expression in 10.5.2 Example 3 for $\boldsymbol{t}$:

$$
\begin{aligned}
\boldsymbol{t}_{\mathrm{S}} &= (\cos\lambda_0 \cos\varphi_0 \quad \sin\lambda_0 \cos\varphi_0 \quad \sin\varphi_0) \\
&= \big(\cos(-77\,\pi/180)\cos(38{,}88\,\pi/180) \quad \sin(-77\,\pi/180)\cos(38{,}88\,\pi/180) \quad \sin(38{,}88\,\pi/180)\big) \\
&= (0{,}175\ 115\ 92 \quad -0{,}758\ 510\ 36 \quad 0{,}627\ 691\ 36).
\end{aligned}
$$

So that the direction vector in SRF GEOCENTRIC_WGS_1984 is $\boldsymbol{n}_{\mathrm{T}} = (0{,}175\ 115\ 92 \quad -0{,}758\ 510\ 36 \quad 0{,}627\ 691\ 36)$.

The case of changing an abstract space linear SRF direction vector $\boldsymbol{n}_{\mathrm{S}}$ to a direction vector $\boldsymbol{n}_{\mathrm{T}}$ in a linear SRF in the space of another object is based on Equation (17). In the notation of 10.4.6:

$$
\boldsymbol{n}_{\mathrm{T}} = \frac{1}{|\boldsymbol{W}|}\boldsymbol{W}\boldsymbol{R}_{\mathrm{S}}\boldsymbol{n}_{\mathrm{S}}\,. \tag{21}
$$

Since a direction vector is a unit vector, division by the determinant cancels any scaling by matrix $\boldsymbol{W}$. $\boldsymbol{R}_{\mathrm{S}}$ is a rotation matrix and therefore its determinant is 1.

EXAMPLE 2 In ISO/IEC 18023-1 (see I18023-1), if an instance of the class <DRM Geometry Model Instance> has a component of class <DRM World Transformation>, that component specifies an invertible matrix $\boldsymbol{W}$ and a coordinate $\boldsymbol{c}$ in the <DRM Environment Root> SRF. If $\boldsymbol{c}_{\mathrm{S}}$ and $\boldsymbol{n}_{\mathrm{S}}$ are a reference coordinate and a direction vector in an associated LOCAL_SPACE_RECTANGULAR_3D <DRM Geometry Model>, and $\mathrm{SRF_T}$ is the local tangent frame at $\boldsymbol{c}$, then Equation (17) and Equation (21) may be used to compute $\boldsymbol{c}_{\mathrm{T}}$ and $\boldsymbol{n}_{\mathrm{T}}$, respectively. The methods of 10.4.3 may be used to further change $\boldsymbol{c}_{\mathrm{T}}$ from $\mathrm{SRF_T}$ to the <DRM Environment Root> SRF. This procedure to change <DRM Geometry Model> coordinates and directions to the environment root SRF is termed "model instancing".

10.6 Euclidean distance

This International Standard supports an operation to return the Euclidean distance between two object-space locations using the coordinates of those locations in an SRF.

If $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ are two coordinates in an SRF, and if $\boldsymbol{G}$ is the generating function of the CS of the SRF, the *Euclidean distance* d_{E} between the corresponding points in object-space is given by:

$$
d_{\mathrm{E}}(\boldsymbol{c}_1,\boldsymbol{c}_2) = d\big(\boldsymbol{G}(\boldsymbol{c}_1),\boldsymbol{G}(\boldsymbol{c}_2)\big)
$$

where d is the Euclidean metric.

10.7 Geodesic distance and azimuth on an oblate ellipsoid

10.7.1 Introduction

This International Standard supports the *geodesic distance* and azimuth operations for SRFs that have ellipsoidal height designated as the vertical coordinate-component (see 8.4). These SRFs include those based on SRFT CELESTIODETIC, PLANETODETIC, and all map projection SRFTs.

The zero vertical coordinate-component surface for such an SRF is an oblate ellipsoid. Two distinct points on the surface of the oblate ellipsoid are connected by a surface curve called a geodesic as defined in A.7.3. The distance along the curve between the two points is called the geodesic distance. At each point, the angle between the geodesic and the meridian at the point as defined in 5.8.3.4 is the azimuth at the point with respect to the other point. The operations to return the geodesic distance and azimuths given the surface coordinates of the points are supported in the API.

10.7.2 Geodesic distance

For an oblate ellipsoid, a geodesic does not, in general, lie completely in any single plane [RAPP1] [RAPP2]. If (λ_1, φ_1) and (λ_2, φ_2) are the surface celestiodetic coordinates of two points lying on the oblate ellipsoid, $d_{\mathrm{G}}((\lambda_1, \varphi_1), (\lambda_2, \varphi_2))$ denotes the geodesic distance between the points. On a sphere, the value of $d_{\mathrm{G}}((\lambda_1, \varphi_1), (\lambda_2, \varphi_2))$ may be computed using the methods of spherical trigonometry. However, in the general case of an oblate ellipsoid, closed form solutions typically involve elliptic integrals that usually require numerical approximation. Solutions may also be approximated using a variety of techniques including iterative algorithms and/or numerical methods for systems of differential equations (see [RAPP2]).

In the general case, two surface coordinates $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ are converted to celestiodetic coordinates using the operations defined in 10.4.4. In particular, in the case of a map projection SRF, if $\boldsymbol{Q}$ denotes the inverse mapping equations for the SRF, then:

$$d_{\mathrm{G}} = d_{\mathrm{G}}\left(\boldsymbol{Q}\left(\boldsymbol{c}_1\right), \boldsymbol{Q}\left(\boldsymbol{c}_2\right)\right).$$

NOTE The development of approximation equations for d_{G} has been the subject of much research. There are approximation formulas for the short distance case where d_{G} ≤ 200 km, for the medium distance case where d_{G} ≤ 1000 km and for the long lines case where the points are antipodal or near antipodal. Two points on the oblate ellipsoid are exactly antipodal when $|(\lambda_2 - \lambda_1)| = \pi$ and $\varphi_1 = -\varphi_2$. There are also special cases when $\varphi_1 = \varphi_2$. A thorough exposition of geodesic distance approximations is given in [RAPP1] [RAPP2].

10.7.3 Geodetic azimuth

Geodetic azimuth is defined in 5.8.3.4. On a sphere, a geodesic between two points is an arc of a great circle and the problem of computing the angles of a spherical triangle can be solved in closed form. In the general case of an oblate ellipsoid, the problem of computing the angles of an elliptical triangle does not have a closed solution. Several different approximations are commonly used.

NOTE Some algorithms are designed to compute both the geodesic distance and the azimuths associated with two points.

